With these substitutions, we get

$$r = 2uv \cdot \frac{u-v}{u+v};$$

$$a = \frac{2uv}{u+v} \cdot [(u-v)\cos\theta + (u+v)\sin\theta];$$

$$b = (u^2+v^2)\left(\frac{u-v}{u+v}\right)\cos\theta;$$

$$c = \frac{2uv}{u+v} \cdot [(u-v)\cos\theta - (u+v)\sin\theta];$$

$$x = \frac{2uv}{u^2+v^2} \cdot \frac{u-v}{u+v}(v^2\sin\theta + 2uv\cos\theta - u^2\sin\theta);$$

$$y = 2uv\left(\frac{u-v}{u+v}\right)\cos\theta;$$

$$z = \frac{2uv}{u^2+v^2} \quad \frac{u-v}{u+v}(u^2\sin\theta + 2uv\cos\theta - v^2\sin\theta).$$

With these values of $a, b, c, x, y, z$, we get

$$\frac{a+c}{b} = \frac{4uv}{u^2+v^2} = \frac{x+z}{y},$$

whence the result follows.

**4236**. *Proposed by Nguyen Viet Hung.*

Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}}\right).$$

*We received* 15 *solutions. We present* 4 *solutions.*

*Solution 1, by Henry Ricardo.*

Using Bernoulli's inequality, we have

$$1 \le \sqrt[k]{\frac{(k^2+2)^2}{k^4+4}} = \sqrt[k]{1+\frac{4k^2}{k^4+4}} \le 1 + \frac{4k}{k^4+4} < 1 + \frac{4}{k^3}.$$

Thus

$$1 \le \frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}} < 1 + \frac{4}{n}\sum_{k=1}^{n}\frac{1}{k^3} = 1 + O\left(\frac{1}{n}\right),$$

and the desired limit is 1 as $n \to \infty$.

*Solution 2, by Arkady Alt.*

First note that

$$\frac{\left(k^2+2\right)^2}{k^4+4} > 1 \implies \sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} > 1$$

where $k \in \mathbb{N}$. By Bernoulli Inequality,

$$\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} = \left(1+\frac{4k^2}{k^4+4}\right)^{1/k} \le 1+\frac{4k^2}{k^4+4}\cdot\frac{1}{k} = 1+\frac{4k}{k^4+4},$$

where $k \in \mathbb{N}$.

Thus, for any natural $k$ we have the double inequality

$$1 < \sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \le 1+\frac{4k}{k^4+4}$$

and therefore

$$n < \sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \le n+\sum_{k=1}^{n}\frac{4k}{k^4+4}.$$

Noting that

$$\begin{aligned}\frac{4k}{k^4+4} &= \frac{4k}{\left(k^2-2k+2\right)\left(k^2+2k+2\right)}\\ &= \frac{1}{k^2-2k+2}-\frac{1}{k^2+2k+2}\\ &= \frac{1}{(k-1)^2+1}-\frac{1}{(k+1)^2+1}\\ &= b_k-b_{k+1},\end{aligned}$$

where $b_k := \dfrac{1}{(k-1)^2+1}+\dfrac{1}{k^2+1}$, we obtain

$$\sum_{k=1}^{n}\frac{4k}{k^4+4} = \sum_{k=1}^{n}\left(b_k-b_{k+1}\right) = b_1-b_{n+1} = \frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right).$$

Since

$$n < \sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} \le n+\frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right),$$

then

$$1 < \frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}} < 1+\frac{1}{n}\left(\frac{3}{2}-\left(\frac{1}{n^2+1}+\frac{1}{(n+1)^2+1}\right)\right).$$

By the Squeeze Theorem,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}}\right)=1.$$

*Solution 3, by Leonard Giugiuc.*

Let $x_n=\sum_{k=1}^{n}\sqrt[k]{\frac{(k^2+2)^2}{k^4+4}}$ and $y_n=n$ for all $n\geq 1$. By the Stolz-Cesàro Theorem,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt[k]{\frac{\left(k^2+2\right)^2}{k^4+4}}\right)=\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}}=\lim_{n\to\infty}\sqrt[n]{\frac{\left(n^2+2\right)^2}{n^4+4}}.$$

Since $1<\frac{\left(n^2+2\right)^2}{n^4+4}<2$, by the Squeeze Theorem, we get

$$\lim_{n\to\infty}\sqrt[n]{\frac{\left(n^2+2\right)^2}{n^4+4}}=1.$$

*Solution 4, by Michel Bataille.*

For each positive integer $n$, let $u_n=\left(\frac{(n^2+2)^2}{n^4+4}\right)^{1/n}$. We show that the required limit, that is,

$$\lim_{n\to\infty}\frac{u_1+u_2+\cdots+u_n}{n}$$

is 1.
Since $\lim\limits_{n\to\infty}\frac{4n^2}{n^4+4}=0$, we have, as $n\to\infty$,

$$\ln(u_n)=\frac{1}{n}\ln\left(1+\frac{4n^2}{n^4+4}\right)\ \sim\ \frac{1}{n}\cdot\frac{4n^2}{n^4+4}\ \sim\ \frac{4}{n^3}.$$

It clearly follows that $\lim\limits_{n\to\infty}\ln(u_n)=0$ and so $\lim\limits_{n\to\infty}u_n=1$. The Cesàro mean of the sequence $(u_n)$ has the same limit, which means that

$$\lim_{n\to\infty}\frac{u_1+u_2+\cdots+u_n}{n}=1,$$

as claimed.